Home Work VI

1. Let (x_n) be a C-contractive seq. $(0<C<1)$:

$$|x_{n+1}-x_n| \leq C|x_n - x_{n-1}| \quad \forall\, n \geq 2.$$

Show by MI that $|x_{n+1}-x_n| \leq C^{n-1}|x_2-x_1|$ and that $|x_m - x_n| \leq \left(C^{m-2} + \cdots + C^{n-1}\right)|x_2 - x_1| \quad \forall\, m>n$.
Using ε-N definition and $\lim_n C^n = 0$ show hence that (x_n) is Cauchy.

2*. Respectively by MCT and by Q1, show the sequence (x_n) converges, where $x_1 = 99$ and

$$x_{n+1} = \frac{1}{3}(x_n + 10) \quad \forall\, n$$

Find the limit.

3*. Use MCT to show that (y_n) converges; find its limit:

$$y_1 := 81 \quad \text{and} \quad y_{n+1} = \sqrt{y_n} \quad \forall\, n.$$

4. Let (x_n) be a bounded sequence and recall that

$$\limsup_n x_n := \lim_n y_n \; (= l \in \mathbb{R}, \text{ say}),$$

where $y_n = \sup\{x_n, x_{n+1}, x_{n+2}, \ldots\} \;\forall\, n$. Let α, β be real numbers such that

$$\alpha < l < \beta$$

Show that

(i) $\exists\, N \in \mathbb{N}$ s.t.

$$x_n < \beta \quad \forall\, n \geq N$$

(ii) $\forall\, N \in \mathbb{N}$, $\exists\; n \geq N$ s.t.

$$\alpha < x_n$$

5. With $\alpha = l - \frac{1}{k}$ and $\beta = l + \frac{1}{k}$ in Q4, show that $\exists$ a strictly increasing seq (n_k) of natural numbers such that

$$l - \tfrac{1}{k} < x_{n_k} < l + \tfrac{1}{k} \quad \forall\, k \in \mathbb{N}.$$

Show that $\lim_k x_{n_k} = \limsup_n x_n \left(= \lim_n y_n\right)$.

6. Show conversely that if (x_{m_k}) is a convergent subsequence of (x_n) then

$$\lim_k x_{m_k} \leq \limsup_n x_n.$$

7.* Let X consist of all real numbers expressible as the limit of a convergent subsequence of (x_n).

Show that $\max X = \limsup_n x_n$.

Show further that $\min X = \liminf_n x_n$, i.e. $\min X = \lim_n z_n$, where $z_n = \inf\{x_n, x_{n+1}, \cdots\}$.

8^*: Let $0 < x_n$ and $\limsup_n \frac{x_{n+1}}{x_n} = r \in (0,1)$.

Show that $\sum_{n=1}^{\infty} x_n < +\infty$.